Coframe teleparallel models of gravity. Exact solutions.

*Yakov Itin
Institute of Mathematics
Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel

Abstract

The superstring and superbrane theories include gravity as a necessary and fundamental part of a (future) unified field theory. Thus it is important to consider the alternative representations of general relativity as well as the alternative models of gravity.

We study the coframe teleparallel theory of gravity with a most general quadratic Lagrangian. The coframe field on a differentiable manifold is a basic dynamical variable. A metric tensor as well as a metric compatible connection is generated by a coframe in a unique manner.

The Lagrangian is a general linear combination of Weitzenböck's quadratic invariants with free dimensionless parameters ρ_1, ρ_2, ρ_3. Every independent term of the Lagrangian is a global $SO(1,3)$-invariant 4-form. For a special choice of parameters which confirms with the local $SO(1,3)$ invariance this theory gives an alternative description of Einsteinian gravity - teleparallel equivalent of GR.

The field equations of the theory is studied by a "diagonal" coframe ansatz which is a subclass of a most general spherical-symmetric Einstein-Mayer ansatz. The restricted Lagrangian depends only on two free parameters ρ_1, ρ_3.

We obtain a formula for scalar curvature of a pseudo-Riemannian manifold with a metric constructed from the static "diagonal" solution of the field equation. It is proved that the sign of the scalar curvature depends only on a relation between the parameters ρ_1 and ρ_3. Thus by a specific choice of free parameters a manifold of positive or negative curvature can be obtained. The scalar curvature vanishes only for a subclass of models with $\rho_1 = 0$. This subclass includes the teleparallel equivalent of GR.

We obtain the explicit form of all spherically symmetric static solutions of the "diagonal" type to the field equations for an arbitrary choice of free parameters. We prove that the unique asymptotic-flat solution with Newtonian limit is the Schwarzschild solution that holds for a subclass of teleparallel models with $\rho_1 = 0$. Thus the Yang-Mills-type term of the general quadratic coframe Lagrangian should be rejected.

* email itin@math.huji.ac.il

1 Introduction

The resent progress in superstrings/M-branes theories and recognizing the supergravity as a low-energy limit of the supersymmetry renew an interest to alternative models of classical gravity. For instant, the generalization of AdS/CFT correspondence due to Witten and Yau [5] is based on manifolds with a positive scalar curvature. The metric of such manifold can not be obtained as a solution of the vacuum Einstein field equations which gives only the manifolds with zero scalar curvature. We will show in the sequel that the manifolds with a fixed sign of the scalar curvature are generated by generic solutions in teleparallel theories.
The teleparallel description of gravity has been studied for a long time. The pioneer works of Cartan [1], Weitzenböck [4] and Einstein [2] dealt meaningful with the various models of unified (gravity-electromagnetic) field theory.
The investigation in gauge field theory of gravity and in the Einstein-Cartan gravity (see [6], [7],[8] and the references) evoked considerable interest in teleparallel geometry. In the most general metric-affine theory of gravity (MAG) [9] the teleparallel Lagrangian appears as a separated part of the general MAG Lagrangian.
For the recent investigations in this area see Refs. [10] to [20].
The teleparallel approach to gravity can be viewed as a generalization of the classical Einstein theory. The main properties of this framework are as follows:

1. The primary field variable of the theory (i.e. the coframe) has an intrinsic geometric sense. The basic geometrical quantities: metric, metric compatible connection and volume element can be constructed from the coframe in a unique manner.

2. A covariant quadratic Lagrangian of the Yang-Mills type is defined in this teleparallel approach in contrast to classical general relativity (GR).

3. A local energy-momentum current can be defined in a covariant manner.

4. There exists a free-parametric wide class of possible field equations. So in order to get uniqueness additional symmetric requirements may be applied. For instance, the requirement of a symmetry under the group of local $SO(1,3)$ pseudo-rotations uniquely yields the teleparallel equivalent of GR.

5. The origin of gravity in this teleparallel approach is the energy-momentum current as well as the antisymmetric spinorial current.

6. The more general modifications of the classical theory of gravity such as metric-affine gravity (MAG) include the teleparallel Lagrangian as a self-consistent sector.

Let us give a brief account of teleparallel geometry. Consider an n-dimensional differential manifold M endowed with a smooth coframe field $\{\vartheta^a,\ a=0,...,n-1\}$. The 1-forms ϑ^a is declared to be pseudo-orthonormal. This assumption completely determines a metric on the manifold M by the relation

$$g=\eta_{ab}\vartheta^a\otimes\vartheta^b, \tag{1.1}$$

where $\eta_{ab} = (+1, -1, ..., -1)$. And conversely, the metric g is a unique metric that makes the coframe ϑ^a pseudo-orthonormal. Thus it is possible to consider the coframe field ϑ^a as a basic dynamical variable and to treat the metric g as only a secondary structure. In order to have an isotropic structure (without peculiar directions) this coframe variable should be defined only up to *global* pseudo-rotations. In fact the true dynamical variable is the equivalence class of coframes with the pseudo-rotation as an equivalence relation. Thus in additional to invariance under diffeomorphic transformations of the manifold M the basic geometric structure has to be global $SO(1, n-1)$ invariant. Note that the coframe field is a complex of n^2 independent variables in every point of M while the symmetric metric tensor field has only $n(n+1)/2$ independent components.
An additional requirement of *local* $SO(1, n-1)$ invariance restricts the set of n^2 independent variables to a subset of $n(n+1)/2$ variables, which can be related with the independent components of the metric. In this case the coframe (teleparallel) structure coincides with the metric structure. One can still describe this metric geometry by coframes, but in this case it does not represent a different geometrical structure but provides only a different mathematical tool - Cartan's "Repeŕe Mobile".
Thus teleparallel geometry can be considered as a certain generalization of the Riemannian or pseudo-Riemannian metrical geometry. In order to have a simplest (pure) teleparallel structure one can choose the connections on the manifold M to be metric compatible. This metric compatibility of the connection is equivalent to the coframe compatibility:

$$Dg = 0 \qquad <==> \qquad D\vartheta^a = 0, \tag{1.2}$$

where D is the covariant exterior derivative.
Thus a 1-form of connection $w^a{}_b$ is unique defined by the metric g and also, via the relation (1.1), by the coframe field ϑ^a. Therefore the coframe field on the differential manifold completely defines a certain geometrical structure - *coframe structure*, which can be written symbolically as

$$\{M, \vartheta^a, g(\vartheta^a), w^a{}_b(\vartheta^a)\}. \tag{1.3}$$

A straightforward generalization of this pure coframe structure can be obtained by considering the connection $w^a{}_b$ as an additional independent primary variable. Such generalization, which we write symbolically as

$$\{M, \vartheta^a, g(\vartheta^a), w^a{}_b\}. \tag{1.4}$$

can be refereed to as Cartan teleparallel geometry.
The structure can further be generalized by taking the metric field g as an independent geometrical variable:

$$\{M, \vartheta^a, g, w^a{}_b\}. \tag{1.5}$$

This most general geometrical structure is realized in the framework of metric-affine gravity (MAG) [9].
In the present work we study the general teleparallel model of the type (1.3) with a quadratic Lagrangian on a 4D-manifold of Lorentzian signature $\{1, -1, -1, -1\}$.

2 Quadratic Lagrangian

The general quadratic (in the first derivatives of the coframe field ϑ^a) Lagrangian can be written as a linear combination of the Weitzenböck second order invariants [4]. This Lagrangian is known as a translation invariant Lagrangian of Rumpf [26] :

$$\mathcal{L} = \frac{1}{2\ell^2}\sum_{i=1}^{3} a_i (d\vartheta^a \wedge *^{(i)} d\vartheta_a), \tag{2.1}$$

where ℓ is the Plank length and $*$ is the Hodge dual operator. [1]
The Lagrangian includes the strength $d\vartheta^a$ and its irreducible (under the global Lorentz group) parts ${}^{(i)}d\vartheta_a$. The explicit expressions for these irreducible parts are:

$$\begin{aligned} {}^{(1)}d\vartheta^a &= d\vartheta^a - {}^{(2)}d\vartheta^a - {}^{(3)}d\vartheta^a, && (2.2)\\ {}^{(2)}d\vartheta^a &= \frac{1}{3}\vartheta^a \wedge (e_b \rfloor d\vartheta^b), && (2.3)\\ {}^{(3)}d\vartheta^a &= \frac{1}{3}e^a \rfloor (\vartheta^b \wedge d\vartheta_b). && (2.4) \end{aligned}$$

The coefficients a_1, a_2, a_3 in (2.1) are dimensionless constants. Rearranging the different parts of the Lagrangian (2.1) results in a linear combination of quadratic terms

$$\mathcal{L} = \frac{1}{2\ell^2}\sum_{i=1}^{3} \rho_i \,{}^{(i)}V \tag{2.5}$$

with

$$\begin{aligned} {}^{(1)}V &= d\vartheta^a \wedge * d\vartheta_a, && (2.6)\\ {}^{(2)}V &= \Big(d\vartheta_a \wedge \vartheta^a\Big) \wedge *\Big(d\vartheta_b \wedge \vartheta^b\Big), && (2.7)\\ {}^{(3)}V &= \Big(d\vartheta_a \wedge \vartheta^b\Big) \wedge *\Big(d\vartheta_b \wedge \vartheta^a\Big). && (2.8) \end{aligned}$$

The dimensionless coefficients a_i and ρ_i are related [18] as:

$$\rho_1 = \frac{1}{3}\,(a_2 + 2a_1)\ , \qquad \rho_2 = \frac{1}{3}\,(a_3 - a_1)\ , \qquad \rho_3 = \frac{1}{3}\,(a_1 - a_2)\ , \tag{2.9}$$

$$a_1 = \rho_1 + \rho_3\ , \qquad a_2 = \rho_1 - 2\rho_3\ , \qquad a_3 = \rho_1 + 3\rho_2 + \rho_3\ . \tag{2.10}$$

Every independent part of the Lagrangians (2.1) and (2.5) is diffeomorphic covariant and invariant under the global (rigid) Lorentz group - translation invariant. Thus different choices of the free parameters α_i or ρ_i yield different translation invariant models of gravity.
Let us list the a_i and the ρ_i coefficients for different teleparallel Lagrangians [18]:

[1] Here and later the down-indexed coframe is obtained by the Lorentzian metric, thus $\vartheta^0 = \vartheta_0,\ \vartheta^i = -\vartheta_i$, where $i = 1, 2, 3$.

	$GR_{\parallel}$ [8]	vdH [27]	viable	YM	$YM^{\dagger}$	KI [17]
a_1	1	1	1	1	0	1
a_2	-2	-2	-2	1	3	4
a_3	$-\frac{1}{2}$	1	arb.	1	0	1
ρ_1	0	0	0	1	1	2
ρ_2	$-\frac{1}{2}$	0	arb.	0	0	0
ρ_3	1	1	1	0	-1	-1

Note also the following possibility to modify the Lagrangians (2.1) and (2.5). One can add to them a zeroth part

$$\eta = \frac{1}{4}\Lambda\vartheta^a \wedge *\vartheta_a. \tag{2.11}$$

It is easy to see that this term is an analog of the cosmological Λ-term in Einsteinian gravity. We will not use this term in the sequel.

3 Field equation and ansatz

In order to obtain the field equation for the coframe field ϑ^a one has to calculate the variation of the Lagrangians (2.1) or (2.5) relative to a free variation of the coframe. This procedure is not quite regular because of the Hodge operator $*$, which itself depends on the coframe field ϑ^a and should also be a subject of variation. Thus we have, in general, for an arbitrary form w, a non-commutative relation

$$\delta * w - *\delta w \neq 0. \tag{3.1}$$

The variation of the Lagrangians (2.1), (2.5) can be produced in the following ways:
1) Directly by using the Euler-Lagrange equations [24]

$$d\Big(\frac{\partial V}{\partial d\vartheta^a}\Big) + \frac{\partial V}{\partial \vartheta^a} = 0. \tag{3.2}$$

2) By using the master formula of [18] - the explicit expression for the commutator (3.1).
3) By using a formula [21] for variation of the quadratic Lagrangians of the form $\delta(\alpha \wedge *\beta)$.
All these different methods of variation result in the same second order field equation, that we will write symbolically as

$$\begin{aligned}&\rho_1\Big(2^{(1)}L_a + {}^{(1)}Q_a - 2^{(2)}Q_a\Big) + \rho_2\Big(-2^{(2)}L_a + 2^{(3)}Q_a + {}^{(4)}Q_a - 2^{(5)}Q_a\Big) + \\ &\rho_3\Big(-2^{(3)}L_a + 2^{(6)}Q_a + {}^{(7)}Q_a - 2^{(8)}Q_a\Big) = 0,\end{aligned} \tag{3.3}$$

where the leading part (which includes all the pieces with the second order derivatives) is a linear combination of the terms:

$$\begin{aligned}{}^{(1)}L_a &= d * d\vartheta_a, &(3.4)\\ {}^{(2)}L_a &= \vartheta_a \wedge d * (d\vartheta^b \wedge \vartheta_b), &(3.5)\\ {}^{(3)}L_a &= \vartheta_b \wedge d * (\vartheta_a \wedge d\vartheta^b). &(3.6)\end{aligned}$$

The quadratic part (which involves only the pieces quadratic in the first derivatives) is a combination of the terms

$$
\begin{aligned}
{}^{(1)}Q_a &= e_a\rfloor(d\vartheta^b\wedge *d\vartheta_b), && (3.7)\\
{}^{(2)}Q_a &= (e_a\rfloor d\vartheta^b)\wedge *d\vartheta_b, && (3.8)\\
{}^{(3)}Q_a &= d\vartheta_a\wedge *(d\vartheta^b\wedge\vartheta_b), && (3.9)\\
{}^{(4)}Q_a &= e_a\rfloor\Big(d\vartheta^c\wedge\vartheta_c\wedge *(d\vartheta^b\wedge\vartheta_b)\Big), && (3.10)\\
{}^{(5)}Q_a &= (e_a\rfloor d\vartheta^b)\wedge\vartheta_b\wedge *(d\vartheta^c\wedge\vartheta_c), && (3.11)\\
{}^{(6)}Q_a &= d\vartheta_b\wedge *(\vartheta_a\wedge d\vartheta^b), && (3.12)\\
{}^{(7)}Q_a &= e_a\rfloor\Big(\vartheta_c\wedge d\vartheta^b\wedge *(d\vartheta^c\wedge\vartheta_b)\Big), && (3.13)\\
{}^{(8)}Q_a &= (e_a\rfloor d\vartheta^b)\wedge\vartheta_c\wedge *(d\vartheta^c\wedge\vartheta_b). && (3.14)
\end{aligned}
$$

We are mostly interested in static, spherical symmetric solutions of the equation (3.1). It is well known that a general static spherical symmetric metric can be written as

$$ds^2 = e^{2f}dt^2 - e^{2g}(dx^2+dy^2+dz^2), \qquad (3.15)$$

where $f=f(r)$, $g=g(r)$.
Due to Einstein-Mayer [3] a general static, spherical symmetric coframe has the form

$$\vartheta^0 = e^f dt, \qquad \vartheta^i = wx^i dt + e^g dx^i, \qquad i=1,2,3, \qquad (3.16)$$

where $f=f(r)$, $g=g(r)$ and $w=w(r)$.
We begin the consideration with a simplified "diagonal" ansatz

$$\vartheta^0 = e^{f(x,y,z)}dt, \qquad \vartheta^i = e^{g(x,y,z)}dx^i, \qquad i=1,2,3. \qquad (3.17)$$

and in a sequel we will study the spherical-symmetric "diagonal" solutions to the field equation (3.3). Thus we restrict our attention to a sub-family of solutions with an identically vanishing function w. This restriction is mainly made in order to simplify the calculations. The consideration of a non-restricted spherical-symmetric coframe as well as the study of the physical sense of its "non-diagonal" part will be made in a sequel paper. Note, however, that even the restricted coframe leads to a most general spherical-symmetric metric (3.15). Thus it can be believed that the "non-diagonal" pieces of the coframe can be related to an additional (possible, non-gravity) interaction.
For the "diagonal" coframe (3.17) the following "parallel" relation holds

$$\vartheta_a\wedge d\vartheta^a = 0. \qquad (3.18)$$

This condition is diffeomorphic and global $SO(1,3)$ invariant, thus it can serve as a necessary condition to have a "diagonal" coframe. The relation (3.18) shows that the multiplier of the coefficient ρ_2 in the Lagrangian (2.5) is identically zero. Therefore the field equation (3.3) reduces to:

$$\rho_1\Big(2{}^{(1)}L_a + {}^{(1)}Q_a - 2{}^{(2)}Q_a\Big) + \rho_3\Big(-2{}^{(3)}L_a + 2{}^{(6)}Q_a + {}^{(7)}Q_a - 2{}^{(8)}Q_a\Big) = 0. \qquad (3.19)$$

The coframe (3.17) being substituted (Appendix A) in the field equation (3.19) splits it in the temporal ($a = 0$) and in the spatial ($a = i = 1, 2, 3$) parts. The temporal part takes the form

$$\rho_1\Big(-2\triangle f - 2(\nabla f\cdot\nabla g) + 2(\nabla g)^2 - (\nabla f)^2\Big) * \vartheta^0 + \rho_3\Big(-4\triangle g - 2(\nabla g)^2\Big) * \vartheta^0 = 0, \tag{3.20}$$

which is equivalent to a scalar equation

$$\rho_1\Big(2\triangle f + 2(\nabla f\cdot\nabla g) - 2(\nabla g)^2 + (\nabla f)^2\Big) + 2\rho_3\Big(2\triangle g + (\nabla g)^2\Big) = 0. \tag{3.21}$$

As for the spatial part of the field equation (3.19)

$$\begin{aligned}&\rho_1\bigg(\Big(2\triangle g + 2(\nabla f\cdot\nabla g) - (\nabla f)^2\Big)\eta_{ik} + 2g_{ik} + 2f_i g_k - 2f_i f_k - 2g_i g_k\bigg) * \vartheta^k + \\ &2\rho_3\bigg(\Big(\triangle f + \triangle g + (\nabla f)^2\Big)\eta_{ik} + f_{ik} + g_{ik} + f_i f_k - f_i g_k - g_i f_k - g_i g_k\bigg) * \vartheta^k = 0.\end{aligned} \tag{3.22}$$

The system (3.20) and (3.22) is exhibited in the scalar form by

$$\begin{cases}\rho_1\Big(2\triangle f + 2(\nabla f\cdot\nabla g) - 2(\nabla g)^2 + (\nabla f)^2\Big) + 2\rho_3\Big(2\triangle g + (\nabla g)^2\Big) = 0\\ \rho_1\bigg(\Big(2\triangle g + 2(\nabla f\cdot\nabla g) - (\nabla f)^2\Big)\eta_{ik} + 2g_{ik} + 2f_i g_k - 2f_i f_k - 2g_i g_k\bigg)\\ \quad +2\rho_3\bigg(\Big(\triangle f + \triangle g + (\nabla f)^2\Big)\eta_{ik} + f_{ik} + g_{ik} + f_i f_k - f_i g_k - g_i f_k - g_i g_k\bigg) = 0.\end{cases} \tag{3.23}$$

The last equation is symmetric in respect to transposition of the indices ($i <-> j$). Thus the result is an over-determined system of 11 equations for two functions f and g. This system certainly has a trivial solution $f = g = 0$ which yields a flat Minkowskian metric. The interesting fact is that this over-determined system has also nontrivial solutions for an almost all values of the free parameters ρ_1, ρ_3. The explicit form of these solutions will be obtained in the sequel.

4 Curvature

In this section we establish a formula for scalar curvature of a manifold, which metric is constructed from the solution of the system (3.23). The result does not depend on a specific solution of the field equation. For actual calculation of the scalar curvature we use the formulas [29] for non-vanishing components of Ricci tensor in the case of "diagonal" metrics.
Let the components of the metric tensor be [2]

$$g_{\alpha\alpha} = e_\alpha e^{2F_\alpha}, \qquad g_{\alpha\beta} = 0 \qquad \text{for} \qquad \alpha \neq \beta, \tag{4.1}$$

where $e_\alpha = \pm 1$.
The components of Ricci tensor are:

$$R_{\alpha\beta} = \sum_{\mu\neq\alpha,\beta}\Big(F_{\mu,\beta}F_{\beta,\alpha} + F_{\alpha,\beta}F_{\mu,\alpha} - F_{\mu,\alpha}F_{\mu,\beta} - F_{\mu,\alpha,\beta}\Big) \qquad \text{for} \qquad \alpha \neq \beta, \tag{4.2}$$

[2] In all formulas in this section the summation over repeated indices is not used.

$$\begin{aligned}
R_{\alpha\alpha} &= \sum_{\mu\neq\alpha}\Big[F_{\alpha,\alpha}F_{\mu,\alpha}-F^2_{\mu,\alpha}-F_{\mu,\alpha,\alpha} \\
&+ e_\alpha e_\mu e^{2(F_\alpha-F_\mu)}\Big(F_{\mu,\mu}F_{\alpha,\mu}-F^2_{\alpha,\mu}-F_{\alpha,\mu,\mu}-F_{\alpha,\mu}\sum_{\nu\neq\alpha,\mu}F_{\nu,\mu}\Big)\Big].
\end{aligned} \tag{4.3}$$

For the diagonal coframe (3.17) we have

$$e_0=-1,\quad e_i=1,\qquad F_0=f,\quad F_i=g\quad\text{for}\quad i=1,2,3. \tag{4.4}$$

Hence using (4.3) we obtain

$$\begin{aligned}
R_{00} &= -e^{2(f-g)}\Big(\Big(F_{1,1}F_{0,1}-F^2_{0,1}-F_{0,1,1}-F_{0,1}(F_{2,1}+F_{3,1})\Big)+ \\
&\qquad \Big(F_{2,2}F_{0,2}-F^2_{0,2}-F_{0,2,2}-F_{0,2}(F_{1,2}+F_{3,2})\Big)+ \\
&\qquad \Big(F_{3,3}F_{0,3}-F^2_{0,3}-F_{0,3,3}-F_{0,3}(F_{1,3}+F_{2,3})\Big)\Big) \\
&= e^{2(f-g)}\Big(f_1^2+f_{11}+f_1g_1+f_2^2+f_{22}+f_2g_2+f_3^2+f_{33}+f_3g_3\Big).
\end{aligned}$$

Thus the temporal component of the Ricci tensor is

$$R_{00}=e^{2(f-g)}\Big(\triangle f+(\nabla f)^2+(\nabla f\cdot\nabla g)\Big). \tag{4.5}$$

Using (4.2) we compute

$$\begin{aligned}
R_{11}=&\Big[F_{1,1}F_{0,1}-F^2_{0,1}-F_{0,1,1}+e_1e_0e^{2(F_1-F_0)}\Big(F_{0,0}F_{1,0}-F^2_{1,0}-F_{1,0,0}-F_{1,0}(F_{2,0}+F_{3,0})\Big)\Big]+ \\
&\Big[F_{1,1}F_{2,1}-F^2_{2,1}-F_{2,1,1}+e_1e_2e^{2(F_1-F_2)}\Big(F_{2,2}F_{1,2}-F^2_{1,2}-F_{1,2,2}-F_{1,2}(F_{0,2}+F_{3,2})\Big)\Big]+ \\
&\Big[F_{1,1}F_{3,1}-F^2_{3,1}-F_{3,1,1}+e_1e_3e^{2(F_1-F_3)}\Big(F_{3,3}F_{1,3}-F^2_{1,3}-F_{1,3,3}-F_{1,3}(F_{0,3}+F_{2,3})\Big)\Big] \\
=&\Big[g_1f_1-f_1^2-f_{11}\Big]-\Big[g_{11}+\Big(g_{22}+g_2(f_2+g_2)\Big)\Big]-\Big[g_{11}+\Big(g_{33}+g_3(f_3+g_3)\Big)\Big].
\end{aligned}$$

Thus

$$R_{11}=-f_{11}-g_{11}+2g_1f_1-f_1^2+g_1^2-\triangle g-(\nabla g\cdot\nabla f)-(\nabla g)^2. \tag{4.6}$$

Therefore, the spatial components of the Ricci curvature tensor are

$$R_{ii}=-f_{ii}-g_{ii}+2g_if_i-f_i^2+g_i^2-\triangle g-(\nabla g\cdot\nabla f)-(\nabla g)^2. \tag{4.7}$$

Consequently, the scalar curvature takes the form

$$\boxed{R=-e^{-2g}\Big(\triangle f+2\triangle g+(\nabla f)^2+(\nabla f\cdot\nabla g)+(\nabla g)^2\Big).} \tag{4.8}$$

Note that this formula is valid for the diagonal coframe (3.17) even if it does not satisfy the field equation (3.23). Calculate now the scalar curvature of the pseudo-Riemannian manifold constructed from a solution of the field equation. Taking the trace of the second equation of the system (3.23) and adding the first equation we obtain

$$2\rho_1\Big(2\triangle g+\triangle f+3(\nabla f\cdot\nabla g)\Big)+4\rho_3\Big(2\triangle g+\triangle f+(\nabla f)^2+(\nabla f\cdot\nabla g)+(\nabla g)^2\Big)=0. \quad (4.9)$$

Substituting the second derivatives expression from the formula (4.8) we get

$$Re^{2g}(\rho_1+2\rho_3)=-\rho_1(\nabla f-\nabla g)^2. \quad (4.10)$$

We will see later that in the special case $\rho_1+2\rho_3=0$ the system (3.23) has a non-trivial solution only if $f=g$.
Thus the sign of the scalar curvature depends only on the relation between free parameters ρ_1,ρ_3 and does not changes on the manifold. This sign is an invariant for every particular teleparallel model (particular choice of the parameters ρ_1,ρ_3).
The conclusion is:

Proposition 4.1: *Suppose that $\rho_1+2\rho_3\neq 0$. The "diagonal" solutions of the field equation (3.3) describe the pseudo-Riemannian manifold with the scalar curvature*

$$\boxed{R=-\frac{\rho_1}{\rho_1+2\rho_3}e^{-2g}(\nabla f-\nabla g)^2} \quad (4.11)$$

The sign of the scalar curvature R is opposite to the sign of $\rho_1(\rho_1+2\rho_3)$ and it is an invariant of a particular teleparallel model.
The scalar curvature is zero if and only if $\rho_1=0$. - the teleparallel equivalent of GR.

The case $\rho_1+2\rho_3=0$ will be treated later. Note that the subclass $\rho_1=0$ includes the Einstein theory of gravity in the form of teleparallel equivalent of GR.
We have shown that the scalar curvature of manifold constructed from a "diagonal" solution of a specific teleparallel model has a fixed sign. In order to clarify the result let us recall a similar fact from the geometry of surfaces. Consider, as an example, a 2D-surface with a metric

$$dl^2=e^{2f}(du^2+dv^2), \quad (4.12)$$

where $f=f(u,v)$. Note that for an arbitrary 2D-manifold every metric can be transformed to this conformal invariant form. The Gauss curvature for the metric (4.12) takes the form

$$K=-e^{-2f}\triangle f. \quad (4.13)$$

Consider now a class of metrics that satisfy a "covariant field equation":

$$\triangle f=\rho(\nabla f)^2. \quad (4.14)$$

The Gauss curvature takes now the form

$$K=-\rho\Big(e^{-2f}(\nabla f)^2\Big). \quad (4.15)$$

The term in the brackets is positive, thus the sign of the curvature depends only of a free parameter ρ of the model. Note that (4.15) is closely similar to the formula (4.11). The

curvature vanishes only in the case $\rho = 0$, i.e. f is a harmonic function. This is a 2D Euclidean analog of Einstein gravity.
In fact we have proved this property of having fixed sign only in a case of "diagonal" metric. Our conjecture that this property can be proved for an arbitrary static metric that satisfies the general teleparallel field equation.

5 Spherical-symmetric ansatz

Consider a solution of the field equation which depends only on one radial variable $s = x^2 + y^2 + z^2$.

$$\vartheta^0 = e^{f(s)} dt^0, \qquad \vartheta^i = e^{g(s)} dt^i, \qquad i = 1, 2, 3. \tag{5.1}$$

With this ansatz we reduce the general problem of determining 16 unknown functions (the components of the coframe ϑ^a) of four variables (coordinates) to determining two functions f and g of one radial variable s. We obtain, instead of (3.23), an over-determined system of three second order ODE for two independent variables

$$\begin{cases} \rho_1\Big(2f''s + 3f' + 2f'g's - 2(g')^2 s + (f')^2 s\Big) + 2\rho_3\Big(2g''s + 3g' + (g')^2 s\Big) = 0 \\ \rho_1\Big(2g'' + 2f'g' - 2(f')^2 - 2(g')^2\Big) + 2\rho_3\Big(f'' + g'' + (f')^2 - 2f'g' - (g')^2\Big) = 0 \\ \rho_1\Big(4g''s + 4g' + 4f'g's - 2(f')^2 s\Big) + 2\rho_3\Big(2f''s + 2f' + 2g''s + 2g' + 2(f')^2 s\Big) = 0. \end{cases} \tag{5.2}$$

We will see later that for certain special values of the free parameters ρ_1, ρ_3 these three equations are algebraically dependent and the system (5.2) reduces to a good posed system of two independent equations for two independent variables. For most of the values of free parameters ρ_1, ρ_3 the three equations of (5.2) are independent. Using this fact we obtain an explicit form of exact solutions. Let us make an algebraic transformation of the system (5.2), which allow to decrease the order of the system. The subtraction of the second equation of (5.2) multiplied by $2s$ from the third equation multiplied by 3 results in

$$\rho_1\Big(4g''s + 6g' + 4f'g's - (f')^2 s + 2(g')^2 s\Big) + \\ 2\rho_3\Big(2f''s + 3f' + 2g''s + 3g' + 2(f')^2 s + 2f'g's + (g')^2 s\Big) = 0.$$

The subtraction of the second equation of (5.2) multiplied by $(2s)$ from the third equation results in

$$\rho_1\Big(2g' + (f')^2 s + 2(g')^2 s\Big) + 2\rho_3\Big(f' + g' + 2f'g's + (g')^2 s\Big) = 0.$$

Thus we obtain an equivalent system to (5.2)

$$\begin{cases} \rho_1\Big(2f''s + 3f' + 2f'g's - 2(g')^2 s + (f')^2 s\Big) + 2\rho_3\Big(2g''s + 3g' + (g')^2 s\Big) = 0 \\ \rho_1\Big(4g''s + 6g' + 4f'g's - (f')^2 s + 2(g')^2 s\Big) + \\ \qquad 2\rho_3\Big(2f''s + 3f' + 2g''s + 3g' + 2(f')^2 s + 2f'g's + (g')^2 s\Big) = 0 \\ \rho_1\Big(2g' + (f')^2 s + 2(g')^2 s\Big) + 2\rho_3\Big(f' + g' + 2f'g's + (g')^2 s\Big) = 0 \end{cases} \tag{5.3}$$

We can now decrease the order of the system (5.3) by introducing new variables

$$u = f' s^{\frac{3}{2}}, \qquad v = g' s^{\frac{3}{2}}. \tag{5.4}$$

WE obtain a system of first order ODE

$$\begin{cases} \rho_1\Big(2s^{\frac{3}{2}}u' + 2uv - 2v^2 + u^2\Big) + 2\rho_3\Big(2s^{\frac{3}{2}}v' + v^2\Big) = 0 \\ \rho_1\Big(4s^{\frac{3}{2}}v' + 4uv - u^2 + 2v^2\Big) + 2\rho_3\Big(2s^{\frac{3}{2}}u' + 2s^{\frac{3}{2}}v' + 2u^2 + 2uv + v^2\Big) = 0 \\ \rho_1\Big(2vs^{\frac{1}{2}} + u^2 + 2v^2\Big) + 2\rho_3\Big(us^{\frac{1}{2}} + vs^{\frac{1}{2}} + 2uv + v^2\Big) = 0. \end{cases} \tag{5.5}$$

Note that the first two equations of (5.5) contain the first order derivatives and the quadratic expressions of the "strength" variables u and v. This is a general form of a non-linear (Einstein type) equation in gravity. The third equation of (5.5) is an algebraic relation between the variables u and v and the radial coordinate s.
Solving the first two equations of (5.5) for the derivatives terms results in

$$\begin{cases} (2\rho_3+\rho_1)(\rho_3-\rho_1)\Big(2s^{\frac{3}{2}}u'\Big) = (\rho_1^2 + 2\rho_3\rho_1 - 4\rho_3^2)u^2 - 2(2\rho_3^2 + \rho_1\rho_3 - \rho_1^2)uv - 2\rho_1(\rho_3+\rho_1)v^2 \\ 2(2\rho_3+\rho_1)(\rho_3-\rho_1)\Big(2s^{\frac{3}{2}}v'\Big) = \rho_1(2\rho_3-\rho_1)u^2 + 4\rho_1^2 uv - 2(2\rho_3^2 - 3\rho_1\rho_3 - \rho_1^2)v^2 \\ \rho_1\Big(2vs^{\frac{1}{2}} + u^2 + 2v^2\Big) + 2\rho_3\Big(us^{\frac{1}{2}} + vs^{\frac{1}{2}} + 2uv + v^2\Big) = 0. \end{cases} \tag{5.6}$$

The three equations (5.5) or (5.6) contain the derivatives u', v' so one can hope to obtain an algebraic equations for the functions u and v. In order to obtain such algebraic relation we writhe the second and the third equations of the system (5.5) as

$$(\rho_1 v' + \rho_3 u' + r_3 v') = F s^{-3/2}, \tag{5.7}$$

$$(\rho_1 v + \rho_3 u + r_3 v) = G s^{-1/2}, \tag{5.8}$$

where the quadratic functions of u and v are

$$F = F(u,v) = -\frac{1}{4}\Big((4\rho_3 - \rho_1)u^2 + 4(\rho_1+\rho_3)uv + 2(\rho_1+\rho_3)v^2\Big), \tag{5.9}$$

$$G = G(u,v) = -\frac{1}{2}\Big(\rho_1 u^2 + 4\rho_3 uv + 2(\rho - 1 + \rho_3)v^2\Big). \tag{5.10}$$

Taking the derivative of the (5.8) and comparing it with (5.7) we obtain

$$\frac{\partial G}{\partial u}u' + \frac{\partial G}{\partial v}v' = \Big(F + \frac{1}{2}G\Big)s^{-1}. \tag{5.11}$$

The derivative terms in (5.6) can be written symbolically as

$$u' = \phi_1(u,v)s^{-3/2}, \qquad v' = \phi_2(u,v)s^{-3/2}. \tag{5.12}$$

Substituting this equations in (5.11) we obtain an algebraic equation that steel contains the independent variable s

$$\frac{\partial G}{\partial u}\phi_1 + \frac{\partial G}{\partial v}\phi_2 = \Big(F + \frac{1}{2}G\Big)s^{1/2}. \tag{5.13}$$

Deriving now s from (5.8) and substituting in (5.13) we obtain an algebraic equation for u and v which we are looking for

$$(\rho_1 v + \rho_3 u + \rho_3 v)\Big(\frac{\partial G}{\partial u}\phi_1 + \frac{\partial G}{\partial v}\phi_2\Big) = \Big(F + \frac{1}{2}G\Big)G. \tag{5.14}$$

In order to write this equation in the explicit form we make a use of an algebraic computations program "REDUCE". The result is

$$\rho_1(\rho_1 - \rho_3)(u - v)(\rho_1 v + \rho_3 u + \rho_3 v)\Big(\rho_1 u^2 + 4\rho_3 uv + 2(\rho_1 + \rho_3)v^2\Big) = 0. \tag{5.15}$$

This equation gives us a necessary condition to obtain a solution of the system (5.5) or (5.6).

Proposition 5.1: *In order to have a nontrivial spherical-symmetric static "diagonal" solution of the field equation (3.3) one of the following condition should be satisfied:*

1. *The free parameters satisfy the relation*

$$\rho_1(\rho_1 - \rho_3) = 0. \tag{5.16}$$

2. *Function u is zero thus*

$$\rho_1 + \rho_3 = 0. \tag{5.17}$$

3. *Function v is zero thus*

$$\rho_3 = 0. \tag{5.18}$$

4. *The nonzero functions u and v satisfy the proportional relation*

$$u = \lambda v, \qquad \lambda = \lambda(\rho_1, \rho_3). \tag{5.19}$$

6 Solutions

Now we are able to derive the explicit form of the solutions for the system (5.2) for different values of the free parameters ρ_1, ρ_3. It is useful to write the formula (4.11) for the scalar curvature via the radial functions u and v. Using the relations

$$\nabla f^2 = 4(f')^2 s, \qquad \nabla g^2 = 4(g')^2 s, \qquad \text{and} \qquad \nabla f \cdot \nabla g = 4f'g's.$$

we obtain

$$R \;=\; -\frac{\rho_1}{\rho_1 + 2\rho_3}e^{-2g}(\nabla f - \nabla g)^2 = -4\frac{\rho_1}{\rho_1 + 2\rho_3}e^{-2g}s(f' - g')^2.$$

Thus the scalar curvature is

$$R = -\frac{4}{s^2}\frac{\rho_1}{\rho_1 + 2\rho_3}e^{-2g}(u - v)^2. \tag{6.1}$$

6.1 GR-type Lagrangian: $\rho_1 = 0$

The system (5.5) takes the form

$$\begin{cases} 2s^{\frac{3}{2}}v' + v^2 = 0 \\ 2s^{\frac{3}{2}}u' + 2s^{\frac{3}{2}}v' + 2u^2 + 2uv + v^2 = 0 \\ us^{\frac{1}{2}} + vs^{\frac{1}{2}} + 2uv + v^2 = 0 \end{cases} \tag{6.2}$$

Integrate the first equation to obtain

$$v = -\frac{\sqrt{s}}{c\sqrt{s}+1}, \tag{6.3}$$

where c is a constant of integration.
Substituting this relation in the third equation of the system (6.2), we find

$$u = \frac{cs}{c^2s-1}. \tag{6.4}$$

Substituting the functions (6.3 - 6.4) in the second equation of (6.2) we obtain an identity. Integrate [3] the functions u and v to obtain the ansatz functions ($r = \sqrt{s}$):

$$f = \ln\frac{1-\frac{1}{cr}}{1+\frac{1}{cr}}, \qquad g = 2\ln(1+\frac{1}{cr}). \tag{6.5}$$

By taking the parameter of integration to be inversely proportional to the mass of the central body $c = \frac{2}{m}$ we obtain the coframe field in the form

$$\vartheta^0 \ = \ \frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}dt, \qquad \vartheta^i = (1+\frac{m}{2r})^2dx^i, \qquad i = 1,2,3. \tag{6.6}$$

This coframe field yields the Schwarzschild metric in isotropic coordinates

$$ds^2 = \left(\frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}\right)^2 dt^2 - \left(1+\frac{m}{2r}\right)^4(dx^2+dy^2+dz^2). \tag{6.7}$$

Note that the parameter ρ_2 is not determined via the "diagonal" ansatz. Thus the Schwarzschild metric is a solution for a family of teleparallel field equations which defined by the parameters: $\rho_1 = 0$ and ρ_2, ρ_3 - arbitrary. Note that for the teleparallel equivalent of GR we need beside of zero ρ_1 an additional condition: $\rho_3 + 2\rho_2 = 0$.

6.2 $\rho_1 = \rho_3$

The system (5.5) takes the form

$$\begin{cases} 2s^{\frac{3}{2}}(u' + 2v') + u(u+2v) = 0 \\ 4s^{\frac{3}{2}}(u' + 2v') + (u+2v)(3u+2v) = 0 \\ 2s^{\frac{1}{2}}(u+2v) + (u+2v)^2 = 0. \end{cases} \tag{6.8}$$

[3] The constant of integration on this step can be omitted, because we can still re-scale the coordinates.

Subtracting the first equation (multiplied by 2) from the second one we obtain that the system is satisfied if and only if

$$u + 2v = 0, \qquad ==> \qquad f + 2g = 0. \tag{6.9}$$

Consequently the coframe is

$$\vartheta^0 = e^{-2g(r)}dt, \qquad \vartheta^i = e^{g(r)}dx^i \tag{6.10}$$

and the metric is

$$ds^2 = e^{-4g(r)}dt^2 - e^{2g(r)}(dx^2 + dy^2 + dz^2). \tag{6.11}$$

Thus the solution incorporates an unknown function $g = g(r)$. The scalar curvature of the metric (6.11) is negative for an arbitrary choice of the function $g = g(r)$.

$$R = -3(g')^2 e^{-2g}. \tag{6.12}$$

6.3 $u = 0$ $\quad ==> \quad$ $\rho_1 + \rho_3 = 0$

The equation (5.15) can have a non-trivial solution with a zero function u if and only if the parameters satisfy

$$\rho_1 + \rho_3 = 0. \tag{6.13}$$

This is a case of $YM^\dagger$ Lagrangian

$$\mathcal{L} = d^\dagger\vartheta^a \wedge *d^\dagger\vartheta_a, \tag{6.14}$$

where $d^\dagger$ is a coderivative operator.
The second and the third equations of (5.5) are satisfied identically for zero u and parameters satisfied (6.13). As for the first equation of (5.5) it gives for function v

$$s^{\frac{3}{2}}v' + v^2 = 0. \tag{6.15}$$

The solution of this equation is

$$v = \frac{\sqrt{s}}{c\sqrt{s} - 2} \qquad ==> \qquad g' = \frac{1}{s(c\sqrt{s} - 2)}.$$

Omitting the constant of integration we obtain the solution

$$f = 0, \qquad g = \ln\Big(1 - \frac{2}{cr}\Big) = \ln\Big(1 + \frac{r_0}{r}\Big),$$

where we introduce a new length dimensional parameter r_0.
Thus the correspondent coframe is

$$\vartheta^0 = dt, \qquad \vartheta^i = \Big(1 + \frac{r_0}{r}\Big)dx^i, \qquad i = 1, 2, 3. \tag{6.16}$$

This coframe generates an asymptotically-flat metric

$$ds^2 = dt^2 - \Big(1 + \frac{r_0}{r}\Big)^2 (dx^2 + dy^2 + dz^2). \tag{6.17}$$

Let us first determine the sign of the parameter r_0. The metric (6.17) represents a point-like solution if the correspondent ADM-mass is positive. Rewrite the metric (6.17) in the spherical Schwarzschild-type coordinates (where the circumference of a circle with center at the origin is equal to $2\pi r$). In the spherical coordinates the metric is

$$ds^2 = dt^2 - \Big(1 + \frac{r_0}{r}\Big)^2 (dr^2 + r^2 d\Omega^2).$$

Using now the translation

$$\tilde{r} = r + r_0$$

we obtain the asymptotic-flat metric in the Schwarzschild coordinates

$$ds^2 = dt^2 - \frac{d\tilde{r}^2}{(1 - \frac{r_0}{\tilde{r}})^2} - \tilde{r}^2 d\Omega^2. \tag{6.18}$$

The ADM mass for the metric (6.18) takes the form

$$m := \lim_{\tilde{r} \to \infty} \frac{\tilde{r}}{2} \Big(1 - (1 - \frac{r_0}{\tilde{r}})^2\Big) = r_0. \tag{6.19}$$

Thus by taking the parameter r_0 to be positive we obtain a particle-type solution with a finite positive ADM-mass.
The metric (6.17) is singular at the origin of coordinates $r = 0$ and consequently the metric (6.18) is singular at $\tilde{r} = r_0$. In order to clarify the nature of this singularity compute the scalar curvature of the metric (6.17) via the formula (6.1). The result is

$$R = \frac{r_0^2}{(r_0 + r)^4} = \frac{r_0^2}{\tilde{r}^4}. \tag{6.20}$$

This function is non-zero and regular for all values of r including the origin ($r = 0$).
The proper distance for a radial null geodesic in the metric (6.18) is equal to the proper time and attach the infinity

$$l = t = \int_{r_0}^{\tilde{r}_1} \frac{d\tilde{r}}{1 - \frac{r_0}{\tilde{r}}} \to \infty$$

Thus the point $r = 0$ ($\tilde{r} = r_0$) does not belong to any final part of the space-time.

6.4 $v = 0$ ==> $\rho_3 = 0$

The equation (5.15) has in this case a non-trivial solution only if $\rho_3 = 0$. The system (5.5) yields $u = 0$ thus it does not has a nontrivial solution.

6.5 Proportional solution.

We see from the equation (5.15) that generic solutions u and v satisfy the homogeneous algebraic equation thus they should be proportional one to the other

$$u = \lambda v, \qquad \lambda = \lambda(\rho_1, \rho_3).$$

Rewrite the third equation of (5.5) as

$$\rho_1\Big(2\frac{v}{\sqrt{s}} + (\frac{u}{\sqrt{s}})^2 + 2(\frac{v}{\sqrt{s}})^2\Big) + 2\rho_3\Big(\frac{u}{\sqrt{s}} + \frac{v}{\sqrt{s}} + 2\frac{u}{\sqrt{s}}\frac{v}{\sqrt{s}} + (\frac{v}{\sqrt{s}})^2\Big) = 0.$$

Substituting $u = \lambda v$ we obtain an algebraic linear equation for $\left(\frac{v}{\sqrt{s}}\right)$ with constant coefficients. Thus the general solution of the system should be of the form

$$u = a\sqrt{s}, \qquad v = b\sqrt{s}, \tag{6.21}$$

where a and b are constants that depend on the parameters ρ_1, ρ_3.
Consequently the ansatz functions f and g satisfy [4]

$$f' = us^{-3/2} = \frac{a}{s} \qquad ==> \qquad f = a\ln\frac{s}{r_0^2}, \tag{6.22}$$

$$g' = vs^{-3/2} = \frac{b}{s} \qquad ==> \qquad g = b\ln\frac{s}{r_0^2}. \tag{6.23}$$

The scalar curvature (6.1) for this type of solutions takes the form

$$R = -4\frac{\rho_1}{\rho_1 + 2\rho_3} \cdot \frac{r_0^{4b}(a-b)^2}{s^{4b+2}}. \tag{6.24}$$

Substituting the solutions (6.21) in the system (5.5) we obtain an algebraic system for two scalars a and b:

$$\begin{cases} \rho_1(a + 2ab - 2b^2 + a^2) + 2\rho_3(b + b^2) = 0 \\ \rho_1(2b + 4ab - a^2 + 2b^2) + 2\rho_3(a + b + 2a^2 + 2ab + b^2) = 0 \\ \rho_1(2b + a^2 + 2b^2) + 2\rho_3(a + b + 2ab + b^2) = 0. \end{cases} \tag{6.25}$$

Extracting the second equation from the first and from the third we obtain an equivalent system

$$\begin{cases} \rho_1(a + 2ab - 2b^2 + a^2) + 2\rho_3(b + b^2) = 0 \\ (a+b)\Big(\rho_1(2b - a) + 2\rho_3 a\Big) = 0 \\ a\Big(\rho_1(2b - a) + 2\rho_3 a\Big) = 0. \end{cases} \tag{6.26}$$

The case $a = 0$ and consequently $u = 0$ was treated above thus we obtain from the third equation of this system that the nontrivial solutions should satisfy

$$\rho_1(2b - a) + 2\rho_3 a = 0.$$

[4]By rescaling the coordinates in the coframe and in the line element we can use the same constant of integration in the functions f and g.

Rejecting the case $\rho_1 = 0$ (also treated above) we obtain

$$b = a\frac{\rho_1 - 2\rho_3}{2\rho_1}. \tag{6.27}$$

Substituting this relation in the first equation of (6.26) we obtain

$$(\rho_1 - \rho_3)(3\rho_1 - 2\rho_3)(\rho_1 + 2\rho_3)a = 2\rho_1(\rho_3 - \rho_1)(\rho_1 + 2\rho_3). \tag{6.28}$$

In order to satisfy this equation we have to separate the following three cases:

- $\boldsymbol{\rho_1 = \rho_3}$.

Via (6.27) the constants are connected as

$$a = -2b$$

and the system (6.26) satisfied identically.
Hence the coframe

$$\vartheta^0 = \left(\frac{r}{r_0}\right)^{2b} dt, \qquad \vartheta^i = \left(\frac{r_0}{r}\right)^{b} dx^i \tag{6.29}$$

includes two free parameters r_0 and b. The correspondent line element is

$$ds^2 = \left(\frac{r}{r_0}\right)^{4b} dt^2 - \left(\frac{r_0}{r}\right)^{2b} (dx^2 + dy^2 + dz^2). \tag{6.30}$$

- $\boldsymbol{\rho_1 = -2\rho_3}$.

Via (6.27) the constants in this case are equal

$$a = b$$

and the system (6.26) satisfied identically. Hence the coframe is

$$\vartheta^0 = \left(\frac{r}{r_0}\right)^{a} dt, \qquad \vartheta^i = \left(\frac{r}{r_0}\right)^{a} dx^i. \tag{6.31}$$

The correspondent line element is conformal flat

$$ds^2 = \left(\frac{r}{r_0}\right)^{2a} (dt^2 - dx^2 - dy^2 - dz^2). \tag{6.32}$$

- $\boldsymbol{3\rho_1 = 2\rho_3}$.

The equation (6.28) does not satisfied for any finite value of a thus the field equation does not has a non-trivial solution.

- ***Algebraic general case***.

Consider the generic case

$$\rho_1 \neq \rho_3, \qquad \rho_1 \neq -2\rho_3, \qquad 3\rho_1 \neq 2\rho_3.$$

From the equations (6.28 - 6.27) the value of the constants are

$$a = \frac{2\rho_1}{2\rho_3 - 3\rho_1}, \qquad b = \frac{\rho_1 - 2\rho_3}{2\rho_3 - 3\rho_1}. \tag{6.33}$$

Now we derive by integration the ansatz functions

$$f = a \ln \frac{s}{r_0^2}, \qquad g = b \ln \frac{s}{r_0^2}. \tag{6.34}$$

Thus the coframe is

$$\vartheta^0 = \left(\frac{r}{r_0}\right)^{2a} dt, \qquad \vartheta^i = \left(\frac{r}{r_0}\right)^{2b} dx^i, \qquad i = 1, 2, 3. \tag{6.35}$$

The correspondent metric element is

$$ds^2 = \left(\frac{r}{r_0}\right)^{4a} dt^2 - \left(\frac{r}{r_0}\right)^{4b} (dx^2 + dy^2 + dz^2). \tag{6.36}$$

Note that the parameters a and b are completely defined by Eq. (6.33).
Consider now how these proportional solutions related to the special solutions considered above.
1) $\quad \boldsymbol{\rho_1 = 0}$
We obtain from the equations (6.33)

$$a = 0, \qquad b = -1$$

thus the solution (6.35) takes the form

$$\vartheta^0 = dt, \qquad \vartheta^i = \left(\frac{r_0}{r}\right)^2 dx^i, \qquad i = 1, 2, 3.$$

The correspondent metric is

$$ds^2 = dt^2 - = \left(\frac{r_0}{r}\right)^2 (dr^2 + r^2 d\Omega^2)$$

Using the following change of the radial coordinate

$$r = \frac{r_0^2}{\tilde{r}}$$

we obtain the Minkowskian metric. Thus the Schwarzschild coframe yields a unique non-flat metric.
2) $\quad \boldsymbol{\rho_1 = \rho_3}$
Substating this relain in (6.33) we obtain

$$a = -2, \qquad b = 1$$

and the solution is

$$\vartheta^0 = \left(\frac{r}{r_0}\right)^{-4} dt, \qquad \vartheta^i = \left(\frac{r}{r_0}\right)^2 dx^i, \qquad i = 1, 2, 3.$$

This is a special case of (6.10).
3) $\quad \boldsymbol{\rho_1 + \rho_3 = 0}$
The relation (6.33) gives

$$a = \frac{-}{2}5, \qquad b = \frac{-}{3}5$$

and the solution (6.35) is

$$\vartheta^0 = \left(\frac{r_0}{r}\right)^{\frac{4}{5}} dt, \qquad \vartheta^i = \left(\frac{r_0}{r}\right)^{\frac{6}{5}} dx^i, \qquad i = 1, 2, 3.$$

Thus the field equation has two non-trivial solutions.

3) $\qquad \boldsymbol{\rho_3 = 0}$

The parameters of the cofrane are

$$a = -\frac{2}{3}, \qquad b = -\frac{1}{3}$$

and the solution is

$$\vartheta^0 = \left(\frac{r_0}{r}\right)^{\frac{4}{3}} dt, \qquad \vartheta^i = \left(\frac{r_0}{r}\right)^{\frac{2}{3}} dx^i, \qquad i = 1, 2, 3.$$

Thus for $\rho_3 = 0$ the field equation has an unique solution.

4) $\qquad \boldsymbol{\rho_1 = -2\rho_3}$

The parameters take the values

$$a = -\frac{1}{2}, \qquad b = -\frac{1}{2}$$

and the solution is

$$\vartheta^0 = \frac{r_0}{r} dt, \qquad \vartheta^i = \frac{r_0}{r} dx^i, \qquad i = 1, 2, 3.$$

This is a special case of the conformal flat solution (6.31).

5) $\qquad \boldsymbol{\rho_1 = 2\rho_3}$

The parameters a, b are non-defined and the proportional solution does not exist.

The results of our analyse can be stated in the following theorem:

Theorem 6.1: *The field equation (3.3) has a non-trivial sphrically symmetric static solytion of the "diogonal" type for all values of the free parameters ρ_1, ρ_3 exept of the case*

$$\rho_1 = 2\rho_3.$$

For

$$\rho_1 = \rho_3$$

exists a family of solutions, which depends on an arbitrary function of radial distance.
For

$$\rho_1 = -2\rho_3$$

exists a family af conformal flat solutions, which parameterized by an arbitrary constant.
For

$$\rho_1 + \rho_3 = 0$$

exist two non-trivial solutions. For all remain values of the parameters ρ_1, ρ_3 the solution is unique.
The assymptotyc flat solution exist only for

$$\rho_1 = 0 \qquad or \qquad \rho_1 + \rho_3 = 0.$$

A unique solution wich gives the Newtonian limit exists for

$$\rho_1 = 0$$

and leads to the Schwarzschild metric.

7 Metrics generated by proportional solutions.

In the previous section it was shown that for generic values of free parameters of the theory the field equation (3.3) has a unique non-trivial static spherical-symmetric coframe solution of the form (6.35). This solution generates the metric (6.36). The physical meaning of these metrics require consideration of the topological properties of the manifold where the metrics can be realized.
The camponents of the metric tensor (6.36) have two coordinate singularities: one at the origin of coordinates $r = 0$ and the second at the coordinate infinity $r = \infty$. In order to clearify the nature of these singularities calculate the scalar curvature of the metric (6.36) Using the formula (6.1) we obtain

$$R = -\frac{4\rho_1}{\rho_1 + 2\rho_3}\frac{(a-b)^2}{r_0^2}\Big(\frac{r_0}{r}\Big)^{4b+2} \tag{7.1}$$

Thus in the case

$$2b + 1 > 0 \qquad <==> \qquad \frac{\rho_1}{\rho_3} < -2 \quad \text{or} \quad \frac{\rho_1}{\rho_3} > \frac{2}{3} \tag{7.2}$$

the scalar curvature in the origin $r = 0$ is infinity.
Accordingly in the case

$$2b + 1 > 0 \qquad <==> \qquad -2 < \frac{\rho_1}{\rho_3} < \frac{2}{3} \tag{7.3}$$

the scalar curvature at the second coordinate singularity $r = \infty$ is infinity.
Note also the sign of the scalar curvature:

$$R < 0 \qquad <==> \qquad -2 < \frac{\rho_1}{\rho_3} < 0 \tag{7.4}$$

and

$$R > 0 \qquad <==> \qquad \frac{\rho_1}{\rho_3} > 0 \quad \text{or} \quad \frac{\rho_1}{\rho_3} < -2. \tag{7.5}$$

Let us transform now the metric (6.36) to the Schwarzschild coordinates. In the spherical coordinates this metric takes the form

$$ds^2 = \Big(\frac{r}{r_0}\Big)^{4a} dt^2 - \Big(\frac{r}{r_0}\Big)^{4b}(dr^2 + r^2 d\Omega^2). \tag{7.6}$$

In the case

$$2b+1\neq 0 \qquad <==> \qquad \frac{\rho_1}{\rho_3}\neq -2 \tag{7.7}$$

we can use the transformation

$$\tilde{r}=r\Big(\frac{r}{r_0}\Big)^{2b} \qquad <==> \qquad r=r_0\Big(\frac{\tilde{r}}{r_0}\Big)^{\frac{1}{2b+1}} \tag{7.8}$$

The metric (6.36) in the new (Schwarzschild) coordinates takes the form

$$ds^2=\Big(\frac{\tilde{r}}{r_0}\Big)^{\frac{4a}{2b+1}}dt^2-\frac{dr^2}{(2b+1)^2}-r^2d\Omega^2. \tag{7.9}$$

Note that that the hyper-spaces $t=const$ is not flat, although b is constant. The four dimensional metric (7.9) is spherically symmetric thus for analyzing the nature of the singularity it is enough to consider it's two-dimensional part "$\tilde{r}-t$". The proper radial distance from an arbitrary point r_1 to a singular point r_* is

$$l=\Big|\int_{\tilde{r}_1}^{\tilde{r}_*}\frac{d\tilde{r}}{2b+1}\Big|=\Big|\frac{\tilde{r}_*-\tilde{r}_1}{2b+1}\Big|. \tag{7.10}$$

As for a proper time to attach the singular point r_* by a radial null geodesic is

$$t=\Big|\frac{1}{2b+1}\int_{\tilde{r}_1}^{\tilde{r}_*}\Big(\frac{r_0}{\tilde{r}}\Big)^{\frac{2a}{2b+1}}d\tilde{r}\Big|=\frac{|r_*^m-r_1^m|}{|2b-2a+1|}, \tag{7.11}$$

where

$$m=\frac{2b-2a+1}{2b+1}=\frac{2\rho_3-3\rho_1}{\rho_1+2\rho_3}.$$

Thus we obtain the following regimes for the singularity in the origin of the coordinates:

$\boldsymbol{r=0}$	$\frac{\rho_1}{\rho_3}<-2$	$-2<\frac{\rho_1}{\rho_3}<-\frac{1}{2}$	$-\frac{1}{2}<\frac{\rho_1}{\rho_3}<\frac{2}{3}$	$\frac{\rho_1}{\rho_3}>\frac{2}{3}$
New coordinates	$\tilde{r}=0$	$\tilde{r}=\infty$	$\tilde{r}=\infty$	$\tilde{r}=0$
Scalar curvature	$R=\infty$	$R=0$	$R=0$	$R=\infty$
Proper distance	finite	∞	∞	finite
Parameter m	$m<0$	$m>0$	$m<0$	$m>0$
Proper time	∞	∞	finite	∞

Analyses of this values gives the following regimes:

- $\boldsymbol{\frac{\rho_1}{\rho_3}<-2}$ **and** $\boldsymbol{\frac{\rho_1}{\rho_3}>\frac{2}{3}}$
  For these ranges of the parameters we obtain that in the point $r=0$ the scalar curvature is singular. This point is located at a finite proper distance from an arbitrary point on the manifold but it takes an infinite time to reach this point by a radial null geodesic. Therefore for these values of the free parameters the origin of the coordinates is a physical singular point of the same type as the central point of the Schwarzschild metric.

- $\mathbf{-2 < \frac{\rho_1}{\rho_3} < -\frac{1}{2}}$
  The scalar curvature in the origin of coordinates is zero. This point is located on an infinite proper distance from an arbitrary point on the manifold and it takes an infinity time to reach this point by a null geodesic. Therefore in this case the point $r = 0$ does not belong to any final portion of the space time.

- $\mathbf{-\frac{1}{2} < \frac{\rho_1}{\rho_3} < \frac{2}{3}}$
  The origin $r = 0$ is a point with zero scalar curvature. It is located in an infinity distance from every point but it can be reached by a final time. Thus the singular point is located outside of the null cone.

As for the second coordinate singularity $r = \infty$ we have a following table of regimes:

$\boldsymbol{r = \infty}$	$\frac{\rho_1}{\rho_3} < -2$	$-2 < \frac{\rho_1}{\rho_3} < -\frac{1}{2}$	$-\frac{1}{2} < \frac{\rho_1}{\rho_3} < \frac{2}{3}$	$\frac{\rho_1}{\rho_3} > \frac{2}{3}$
New coordinates	$\tilde{r} = \infty$	$\tilde{r} = 0$	$\tilde{r} = 0$	$\tilde{r} = \infty$
Scalar curvature	$R = 0$	$R = \infty$	$R = \infty$	$R = 0$
Proper distance	∞	finite	finite	∞
Parameter m	$m < 0$	$m > 0$	$m < 0$	$m > 0$
Proper time	finite	finite	∞	∞

Thus we have the following regimes:

- $\mathbf{\frac{\rho_1}{\rho_3} < -2}$
  The scalar singularity in the point $r = \infty$ is zero. This point is located on an infinity distance from every finite point on the manifold. Although it takes a finite time to reach this point from an arbitrary point on the manifold by a radial null geodesic. Thus the point $r = \infty$ is located outside of the null cone.

- $\mathbf{-2 < \frac{\rho_1}{\rho_3} < -\frac{1}{2}}$
  The scalar curvature is infinite thus the point $r = \infty$ is a physical singularity. This point is located on a finite distance from an arbitrary point and can be reached by a finite time. The singularity is similar to a vertex of the cone - conical singularity.

- $\mathbf{-\frac{1}{2} < \frac{\rho_1}{\rho_3} < \frac{2}{3}}$
  For these parameters the point $r = \infty$ is, also, a physical singularity. It can be however be reached by a infinite time nevertheless it is located on a finite proper distance. Thus this singularity is similar to the central point of the Schwarzschild metric.

- $\mathbf{\frac{\rho_1}{\rho_3} > \frac{2}{3}}$
  The proper distance and the proper time is infinite. Therefore the point is located outside of any finite area of space-time.

8 Conclusions and discussion.

We study the general quadratic Lagrangian of pure teleparallel theory. This Lagrangian is a linear combination of three covariant and global $SO(1,3)$ invariant terms with free dimensionless coefficients ρ_1, ρ_2, ρ_3. The field equations of the theory is studied by a "diagonal" ansatz which is a subclass of a general spherical-symmetric Einstein-Mayer ansatz. The restriction is taken for simplification the calculation. Thus we study a Lagrangian that depends only on two free parameters ρ_1, ρ_3.

We obtain a formula for scalar curvature of a pseudo-Riemannian manifold with a metric constructed from the solution of the field equation. This formula shows that the sign of the scalar curvature depends only on a relation between the parameters ρ_1 and ρ_3. The scalar curvature vanishes only in a case $r_1 = 0$ which corresponds to the teleparallel equivalent of GR.

We obtain exact solutions for all possible values of free parameters ρ_1, ρ_3. It is shown that the unique solution with a Newtonian limit is the Schwarzschild solution. Thus the Yang-Mills-type term of the general quadratic Lagrangian should be rejected.

Note that all the results valid only in the case of a restricted "diagonal" ansatz. Thus it is important to obtain solutions for following questions:

- Which physical sense can be given to the various exact solutions?
- How the results depends on the non-diagonal terms?
- How the scalar curvature inequalities can be generalized to the case of three free parameters ρ_1, ρ_2, ρ_3?

Acknowledgments

I am deeply grateful to F.W. Hehl and to S. Kaniel for constant support, interesting discussions and valuable comments.

A Calculations with the diagonal ansatz

We study the field equation (3.3) with the diagonal coframe ansatz

$$\vartheta^0 = e^f dt, \qquad \vartheta^i = e^g dx^i, \qquad i = 1, 2, 3.$$

We will use the following notations

$$(\nabla f \cdot \nabla g) = f_1 g_1 + f_2 g_2 + f_3 g_3 \tag{A.1}$$

$$(\nabla f)^2 = f_1^2 + f_2^2 + f_3^2 \tag{A.2}$$

$$\triangle f = f_{11} + f_{22} + f_{33} \tag{A.3}$$

Calculating the first derivatives of the diagonal coframe we obtain

$d\vartheta^0 = -e^{-g}(f_1\vartheta^{01} + f_2\vartheta^{02} + f_3\vartheta^{03})$	$*d\vartheta^0 = -e^{-g}(f_1\vartheta^{23} - f_2\vartheta^{13} + f_3\vartheta^{12})$
$d\vartheta^1 = -e^{-g}(g_2\vartheta^{12} + g_3\vartheta^{13})$	$*d\vartheta^1 = e^{-g}(g_2\vartheta^{03} - g_3\vartheta^{02})$
$d\vartheta^2 = e^{-g}(g_1\vartheta^{12} - g_3\vartheta^{23})$	$*d\vartheta^2 = -e^{-g}(g_1\vartheta^{03} - g_3\vartheta^{01})$
$d\vartheta^3 = e^{-g}(g_1\vartheta^{13} + g_2\vartheta^{23})$	$*d\vartheta^1 = e^{-g}(g_1\vartheta^{02} - g_2\vartheta^{01})$

A.1 Temporal components

The first leading term of (3.3) takes the form

$$^{(1)}L_0 \quad = \quad d * d\vartheta_0 = -e^{-2g}\Big(\triangle f + (\nabla f \cdot \nabla g)\Big)\vartheta^{123}, \tag{A.4}$$

Thus

$$\boxed{^{(1)}L_0 = -e^{-2g}\Big(\triangle f + (\nabla f \cdot \nabla g)\Big) * \vartheta^0} \tag{A.5}$$

As for the second leading term of (3.3)

$$\begin{aligned}
^{(3)}L_0 &= \vartheta_b \wedge d * (\vartheta_0 \wedge d\vartheta^b) = -\vartheta^1 \wedge d * (\vartheta^0 \wedge d\vartheta^1) - \vartheta^2 \wedge d * (\vartheta^0 \wedge d\vartheta^2) - \vartheta^3 \wedge d * (\vartheta^0 \wedge d\vartheta^3)\\
&= -\vartheta^1 \wedge d * (-e^{-g})(g_2\vartheta^{012} + g_3\vartheta^{013}) - \vartheta^2 \wedge d * (e^{-g})(g_1\vartheta^{012} - g_3\vartheta^{023})\\
&\quad -\vartheta^3 \wedge d * (e^{-g})(g_1\vartheta^{013} + g_2\vartheta^{023})\\
&= \vartheta^1 \wedge d(e^{-g})(g_2\vartheta^3 - g_3\vartheta^2) - \vartheta^2 \wedge d(e^{-g})(g_1\vartheta^3 - g_3\vartheta^1) + \vartheta^3 \wedge d(e^{-g})(g_1\vartheta^2 - g_2\vartheta^1)\\
&= \vartheta^1 \wedge d(g_2 dz - g_3 dy) - \vartheta^2 \wedge d(g_1 dz - g_3 dx) + \vartheta^3 \wedge d(g_1 dy - g_2 dx) = 2e^{-2g}\triangle g \vartheta^{123}
\end{aligned} \tag{A.6}$$

Thus

$$\boxed{^{(3)}L_0 = 2e^{-2g}\triangle g * \vartheta^0} \tag{A.7}$$

The first quadratic term is

$$\begin{aligned}
^{(1)}Q_0 &= e_0 \rfloor (d\vartheta^b \wedge *d\vartheta_b) = e_0 \rfloor \Big(d\vartheta^0 \wedge *d\vartheta^0 - d\vartheta^1 \wedge *d\vartheta^1 - d\vartheta^2 \wedge *d\vartheta^2 - d\vartheta^3 \wedge *d\vartheta^3\Big)\\
&= e^{-2g} e_0 \rfloor \Big((f_1\vartheta^{01} + f_2\vartheta^{02} + f_3\vartheta^{03}) \wedge (f_1\vartheta^{23} - f_2\vartheta^{13} + f_3\vartheta^{12}) - (g_2\vartheta^{12} + g_3\vartheta^{13})(-g_2\vartheta^{03} + g_3\vartheta^{02}) +\\
&\quad (g_1\vartheta^{12} + g_3\vartheta^{23})(-g_1\vartheta^{03} - g_3\vartheta^{01}) - (g_1\vartheta^{13} + g_2\vartheta^{23})(g_1\vartheta^{02} - g_2\vartheta^{01})\\
&= e^{-2g}\Big(\nabla^2 f + 2\nabla^2 g\Big)\vartheta^{123}
\end{aligned} \tag{A.8}$$

Thus

$$\boxed{{}^{(1)}Q_0 = e^{-2g}\Big((\nabla f)^2 + 2(\nabla g)^2\Big) * \vartheta^0} \tag{A.9}$$

The second quadratic term is

$$\begin{aligned} {}^{(2)}Q_0 &= (e_0 \rfloor d\vartheta^b) \wedge * d\vartheta_b = (e_0 \rfloor d\vartheta^0) \wedge * d\vartheta_0 \\ &= e^{-2g}(f_1\vartheta^1 + f_2\vartheta^2 + f_3\vartheta^3) \wedge (f_1\vartheta^{23} - f_2\vartheta^{13} + f_3\vartheta^{12}) = e^{-2g}\nabla^2 f\vartheta^{123} \end{aligned} \tag{A.10}$$

Thus

$$\boxed{{}^{(2)}Q_0 = e^{-2g}(\nabla f)^2 * \vartheta^0} \tag{A.11}$$

The next non-zero quadratic term is

$$\begin{aligned} {}^{(6)}Q_0 &= d\vartheta_b \wedge *(\vartheta_0 \wedge d\vartheta^b) = -d\vartheta^1 \wedge *(\vartheta^0 \wedge d\vartheta^1) - d\vartheta^2 \wedge *(\vartheta^0 \wedge d\vartheta^2) - d\vartheta^3 \wedge *(\vartheta^0 \wedge d\vartheta^3) \\ &= -e^{-2g}\Big((g_2\vartheta^{12} + g_3\vartheta^{13}) \wedge *(g_2\vartheta^{012} + g_3\vartheta^{013}) + (g_1\vartheta^{12} - g_3\vartheta^{23}) \wedge *(g_1\vartheta^{012} - g_3\vartheta^{023}) \\ &\qquad + (g_1\vartheta^{13} + g_2\vartheta^{23}) \wedge *(g_1\vartheta^{013} + g_2\vartheta^{023})\Big) \\ &= -e^{-2g}\Big((g_2\vartheta^{12} + g_3\vartheta^{13}) \wedge (g_2\vartheta^3 - g_3\vartheta^2) + (g_1\vartheta^{12} - g_3\vartheta^{23}) \wedge (g_1\vartheta^3 - g_3\vartheta^1) \\ &\qquad + (g_1\vartheta^{13} + g_2\vartheta^{23}) \wedge (-g_1\vartheta^2 + g_2\vartheta^1)\Big) = -2e^{-2g}\nabla^2 g\vartheta^{123} \end{aligned} \tag{A.12}$$

Thus

$$\boxed{{}^{(6)}Q_0 = -2e^{-2g}(\nabla g)^2 * \vartheta^0} \tag{A.13}$$

As for the next non-zero quadratic term ${}^{(7)}Q_0$ we have

$$\begin{aligned} {}^{(7)}Q_0 &= e_0 \rfloor \Big(\vartheta_c \wedge d\vartheta^b \wedge *(d\vartheta^c \wedge \vartheta_b)\Big) \\ &= 2e_0 \rfloor \Big(-\vartheta^0 \wedge d\vartheta^1 \wedge *(d\vartheta^0 \wedge \vartheta^1) - \vartheta^0 \wedge d\vartheta^2 \wedge *(d\vartheta^0 \wedge \vartheta^2) - \vartheta^0 \wedge d\vartheta^3 \wedge *(d\vartheta^0 \wedge \vartheta^3) \\ &\qquad + \vartheta^1 \wedge d\vartheta^2 \wedge *(d\vartheta^1 \wedge \vartheta^2) + \vartheta^1 \wedge d\vartheta^3 \wedge *(d\vartheta^1 \wedge \vartheta^3) + \vartheta^2 \wedge d\vartheta^3 \wedge *(d\vartheta^2 \wedge \vartheta^3)\Big) \\ &= 2e^{-2g} e_0 \rfloor \Big((g_2\vartheta^{012} + g_3\vartheta^{013}) \wedge (f_2\vartheta^3 - f_3\vartheta^2) + (g_1\vartheta^{012} - g_3\vartheta^{023}) \wedge (f_1\vartheta^3 - f_3\vartheta^1) \\ &\qquad + (g_1\vartheta^{013} + g_2\vartheta^{023}) \wedge (-f_1\vartheta^2 + f_2\vartheta^1) - g_3\vartheta^{123} \wedge g_3\vartheta^0 - g_2\vartheta^{123} \wedge g_2\vartheta^0 - g_1\vartheta^{123} \wedge g_1\vartheta^0\Big) \\ &= 2e^{-2g}(2\nabla f \cdot \nabla g + \nabla^2 g)\vartheta^{123} \end{aligned} \tag{A.14}$$

Thus

$$\boxed{{}^{(7)}Q_0 = 2e^{-2g}\Big(2(\nabla f \cdot \nabla g) + (\nabla g)^2\Big) * \vartheta^0} \tag{A.15}$$

The last non-zero quadratic term ${}^{(8)}Q_0$ is

$$\begin{aligned} {}^{(8)}Q_0 &= (e_0 \rfloor d\vartheta^b) \wedge \vartheta_c \wedge *(d\vartheta^c \wedge \vartheta_b) \\ &= -(e_0 \rfloor d\vartheta^0) \wedge \Big(\vartheta^1 \wedge *(d\vartheta^1 \wedge \vartheta^0) + \vartheta^2 \wedge *(d\vartheta^2 \wedge \vartheta^0) + \vartheta^3 \wedge *(d\vartheta^3 \wedge \vartheta^0)\Big) \\ &= -e^{-2g}(f_1\vartheta^1 + f_2\vartheta^2 + f_3\vartheta^3) \wedge \Big(\vartheta^1 \wedge *(g_2\vartheta^{012} + g_3\vartheta^{013}) - \vartheta^2 \wedge *(g_1\vartheta^{012} - g_3\vartheta^{023}) - \\ &\qquad \vartheta^3 \wedge *(g_1\vartheta^{013} + g_2\vartheta^{023})\Big) \\ &= -e^{-2g}(f_1\vartheta^1 + f_2\vartheta^2 + f_3\vartheta^3) \wedge \Big(g_2\vartheta^{13} - g_3\vartheta^{12} - g_1\vartheta^{23} + g_3\vartheta^{21} + g_1\vartheta^{32} - g_2\vartheta^{31}\Big) \\ &= 2e^{-2g}(\nabla f \cdot \nabla g)\vartheta^{123} \end{aligned} \tag{A.16}$$

Thus

$$\boxed{{}^{(8)}Q_0 = 2e^{-2g}(\nabla f \cdot \nabla g) * \vartheta^0} \tag{A.17}$$

A.2 Spatial components

The first leading term takes the form

$$\begin{aligned}
{}^{(1)}L_1 &= d * d\vartheta_1 = d\Big(e^f(-g_2 dt\wedge dz + g_3 dt\wedge dy)\Big)\\
&= -e^{-2g}\Big((g_{12}+f_1g_2)\vartheta^{013} + (g_{22}+g_{33}+f_2g_2+f_3g_3)\vartheta^{023} - (g_{13}+f_1g_3)\vartheta^{012}\Big)\\
&= e^{-2g}\Big((\triangle g + \nabla f\cdot\nabla g - g_{11} - f_1g_1) * \vartheta^1 - (g_{12}+f_1g_2) * \vartheta^2 - (g_{13}+f_1g_3) * \vartheta^3\Big) \qquad \text{(A.18)}
\end{aligned}$$

Thus

$$\boxed{{}^{(1)}L_i = -e^{-2g}\Big(\Big(\triangle g + (\nabla f\cdot\nabla g)\Big)\eta_{ik} + g_{ik} + f_ig_k\Big) * \vartheta^k} \qquad \text{(A.19)}$$

The second leading term is

$$\begin{aligned}
{}^{(3)}L_1 &= \vartheta_b \wedge d * (\vartheta_1 \wedge d\vartheta^b)\\
&= -\vartheta^0 \wedge d * (\vartheta^1\wedge d\vartheta^0) + \vartheta^2 \wedge d * (\vartheta^1 \wedge d\vartheta^2) + \vartheta^3\wedge d*(\vartheta^1\wedge d\vartheta^3)\\
&= -\vartheta^0\wedge d*(e^{-g})(f_2\vartheta^{012}+f_3\vartheta^{013}) - \vartheta^2\wedge d*(e^{-g})g_3\vartheta^{123}) + \vartheta^3\wedge d*(e^{-g})g_2\vartheta^{123})\\
&= -\vartheta^0\wedge d(e^{-g})(f_2\vartheta^3 - f_3\vartheta^2) - \vartheta^2\wedge d(e^{-g}g_3\vartheta^0) + \vartheta^3\wedge d(e^{-g}g_2\vartheta^0)\\
&= -\vartheta^0\wedge d(f_2dz - f_3dy) - \vartheta^2\wedge d(e^{f-g}g_3dt) + \vartheta^3\wedge d(e^{f-g}g_2dt)\\
&= -\vartheta^0\wedge\Big((f_{22}+f_{33})dy\wedge dz + f_{12}dx\wedge dz - f_{13}dx\wedge dy\Big)\\
&\quad -\vartheta^2\wedge e^{f-g}\Big([g_{13}+(f_1-g_1)g_3]dx\wedge dt + [g_{33}+(f_3-g_3)g_3]dz\wedge dt\Big)\\
&\quad +\vartheta^3\wedge e^{f-g}\Big([g_{12}+(f_1-g_1)g_2]dx\wedge dt + [g_{22}+(f_2-g_2)g_2]dy\wedge dt\Big)\\
&= e^{-2g}\Big((-\triangle f - \triangle g + (\nabla g)^2 - \nabla f\cdot\nabla g + f_{11} + g_{11} - g_1^2 + f_1g_1) * \vartheta^1\\
&\quad +(f_{12}+g_{12}+f_1g_2-g_1g_2) * \vartheta^2 + (f_{13}+g_{13}+f_1g_3-g_1g_3) * \vartheta^3\Big) \qquad \text{(A.20)}
\end{aligned}$$

Thus

$$\boxed{{}^{(3)}L_i = e^{-2g}\Big(\Big(\triangle f + \triangle g - (\nabla g)^2 + (\nabla f\cdot\nabla g)\Big)\eta_{ik} + f_{ik} + g_{ik} - g_ig_k + f_ig_k\Big) * \vartheta^k} \qquad \text{(A.21)}$$

Using the calculations of ${}^{(7)}Q_0$ we obtain the first quadratic term as

$$\boxed{{}^{(1)}Q_i = e_1\rfloor(d\vartheta^b\wedge * d\vartheta_b) = e^{-2g}\Big((\nabla f)^2 + 2(\nabla g)^2\Big)\eta_{ik} * \vartheta^k} \qquad \text{(A.22)}$$

As for the second quadratic term

$$\begin{aligned}
{}^{(2)}Q_1 &= (e_1\rfloor d\vartheta^b)\wedge * d\vartheta_b = (e_1\rfloor d\vartheta^0)\wedge * d\vartheta^0 - (e_1\rfloor d\vartheta^1)\wedge * d\vartheta^1 - (e_1\rfloor d\vartheta^2)\wedge * d\vartheta^2 - (e_1\rfloor d\vartheta^3)\wedge * d\vartheta^3\\
&= e^{-2g}\Big(-\vartheta^0\wedge(f_1\vartheta^{23} - f_2\vartheta^{13} + f_3\vartheta^{12}) - (g_2\vartheta^2 + g_3\vartheta^3)\wedge(-g_2\vartheta^{03} + g_3\vartheta^{02})\\
&\quad -g_1\vartheta^2\wedge(g_1\vartheta^{03} - g_3\vartheta^{01}) - g_1\vartheta^3\wedge(g_1\vartheta 02 - g_2\vartheta^{01})\Big)\\
&= e^{-2g}\Big(-(f_1^2 + g_2^2 + g_3^2 + 2g_1^2)\vartheta^{023} + (f_1f_2 + g_1g_2)\vartheta^{013} - (f_1f_3+g_1g_3)\vartheta^{012}\Big) \qquad \text{(A.23)}
\end{aligned}$$

Thus

$$\boxed{{}^{(2)}Q_i = e^{-2g}\Big((\nabla g)^2\eta_{ik} - (f_if_k + g_ig_k)\Big) * \vartheta^k} \qquad \text{(A.24)}$$

The next non-zero quadratic term is

$$
\begin{aligned}
{}^{(6)}Q_1 &= d\vartheta_b \wedge *(\vartheta_1 \wedge d\vartheta^b) = -d\vartheta^0 \wedge *(\vartheta^1 \wedge d\vartheta^0) + d\vartheta^2 \wedge *(\vartheta^1 \wedge d\vartheta^2) + d\vartheta^3 \wedge *(\vartheta^1 \wedge d\vartheta^3) \\
&= e^{-2g}\Big((f_1\vartheta^{01} + f_2\vartheta^{02} + f_3\vartheta^{03}) \wedge *(f_2\vartheta^{012} + f_3\vartheta^{013}) - \\
&\qquad (g_1\vartheta^{12} - g_3\vartheta^{23}) \wedge *g_3\vartheta^{123} + (g_1\vartheta^{13} + g_2\vartheta^{23}) \wedge *g_2\vartheta^{123}\Big) \\
&= e^{-2g}\Big((f_1\vartheta^{01} + f_2\vartheta^{02} + f_3\vartheta^{03}) \wedge (f_2\vartheta^3 - f_3\vartheta^2) - (g_1\vartheta^{12} - g_3\vartheta^{23}) \wedge g_3\vartheta^0 + (g_1\vartheta^{13} + g_2\vartheta^{23}) \wedge *g_2\vartheta^0\Big) \\
&= e^{-2g}\Big((f_2^2 + f_3^2 + g_2^2 + g_3^2)\vartheta^{023} + (f_1f_2 + g_1g_2)\vartheta^{013} - (f_1f_3 + g_1g_3)\vartheta^{012}\Big) \qquad \text{(A.25)}
\end{aligned}
$$

Thus

$$
\boxed{{}^{(6)}Q_i = -e^{-2g}\Big(\big((\nabla f)^2 + (\nabla g)^2\big)\eta_{ik} + f_if_k + g_ig_k\Big) * \vartheta^k} \qquad \text{(A.26)}
$$

Using the calculations above for ${}^{(7)}Q_0$ we obtain for the next non-zero quadratic term

$$
{}^{(7)}Q_1 = e_1 \rfloor \Big(\vartheta_c \wedge d\vartheta^b \wedge *(d\vartheta^c \wedge \vartheta_b)\Big) = -2e^{-2g}(2\nabla f \cdot \nabla g + \nabla^2 g) * \vartheta^1
$$

Thus

$$
\boxed{{}^{(7)}Q_i = 2e^{-2g}\Big(2(\nabla f \cdot \nabla g) + (\nabla g)^2\Big)\eta_{ik} * \vartheta^k} \qquad \text{(A.27)}
$$

As for the last non-zero quadratic term

$$
\begin{aligned}
{}^{(8)}Q_0 &= (e_1 \rfloor d\vartheta^b) \wedge \vartheta_c \wedge *(d\vartheta^c \wedge \vartheta_b) \\
&= -(e_1 \rfloor d\vartheta^0) \wedge \Big(\vartheta^1 \wedge *(d\vartheta^1 \wedge \vartheta^0) + \vartheta^2 \wedge *(d\vartheta^2 \wedge \vartheta^0) + \vartheta^3 \wedge *(d\vartheta^3 \wedge \vartheta^0)\Big) \\
&\quad +(e_1 \rfloor d\vartheta^1) \wedge \Big(-\vartheta^0 \wedge *(d\vartheta^0 \wedge \vartheta^1) + \vartheta^2 \wedge *(d\vartheta^2 \wedge \vartheta^1) + \vartheta^3 \wedge *(d\vartheta^3 \wedge \vartheta^1)\Big) \\
&\quad +(e_1 \rfloor d\vartheta^2) \wedge \Big(-\vartheta^0 \wedge *(d\vartheta^0 \wedge \vartheta^2) + \vartheta^1 \wedge *(d\vartheta^1 \wedge \vartheta^2) + \vartheta^3 \wedge *(d\vartheta^3 \wedge \vartheta^2)\Big) \\
&\quad +(e_1 \rfloor d\vartheta^3) \wedge \Big(-\vartheta^0 \wedge *(d\vartheta^0 \wedge \vartheta^3) + \vartheta^1 \wedge *(d\vartheta^1 \wedge \vartheta^3) + \vartheta^2 \wedge *(d\vartheta^2 \wedge \vartheta^3)\Big) \\
&= e^{-2g}\Big(f_1\vartheta^0 \wedge \Big(\vartheta^1 \wedge *(g_2\vartheta^{012} + g_3\vartheta^{013}) - \vartheta^2 \wedge *(g_1\vartheta^{012} - g_3\vartheta^{023}) - \vartheta^3 \wedge *(g_1\vartheta^{013} + g_2\vartheta^{023})\Big) \\
&\qquad +(g_2\vartheta^2 + g_3\vartheta^3) \wedge \Big(\vartheta^0 \wedge *(f_2\vartheta^{012} + f_3\vartheta^{013}) + \vartheta^2 \wedge *g_3\vartheta^{123} + \vartheta^3 \wedge *g_2\vartheta^{123}\Big) \\
&\qquad +g_1\vartheta^2 \wedge \Big(\vartheta^0 \wedge *(f_1\vartheta^{012} - f_3\vartheta^{023}) + \vartheta^1 \wedge *g_3\vartheta^{123} - \vartheta^3 \wedge *g_3\vartheta^{123}\Big) \\
&\qquad +g_1\vartheta^3 \wedge \Big(\vartheta^0 \wedge *(f_1\vartheta^{013} + f_2\vartheta^{023}) - \vartheta^1 \wedge *g_2\vartheta^{123} + \vartheta^2 \wedge *g_1\vartheta^{123}\Big)\Big) \\
&= e^{-2g}\Big(f_1\vartheta^0 \wedge \Big(\vartheta^1 \wedge (g_2\vartheta^3 - g_3\vartheta^2) - \vartheta^2 \wedge (g_1\vartheta^3 - g_3\vartheta^1) + \vartheta^3 \wedge (g_1\vartheta^2 - g_2\vartheta^1)\Big) \\
&\qquad +(g_2\vartheta^2 + g_3\vartheta^3) \wedge \Big(\vartheta^0 \wedge (f_2\vartheta^3 - f_3\vartheta^2) + \vartheta^2 \wedge g_3\vartheta^0 + \vartheta^3 \wedge g_2\vartheta^0\Big) \\
&\qquad +g_1\vartheta^2 \wedge \Big(\vartheta^0 \wedge (f_1\vartheta^3 - f_3\vartheta^1) + \vartheta^1 \wedge g_3\vartheta^0 - \vartheta^3 \wedge g_3\vartheta^0\Big) \\
&\qquad +g_1\vartheta^3 \wedge \Big(\vartheta^0 \wedge (-f_1\vartheta^2 + f_2\vartheta^1) - \vartheta^1 \wedge g_2\vartheta^0 + \vartheta^2 \wedge g_1\vartheta^0\Big)\Big) \\
&= e^{-2g}\Big(-(4f_1g_1 + f_2g_2 + f_3g_3 + 2g_1^2 + g_2^2 + g_3^2)\vartheta^{023} + (2f_1g_2 + g_1f_2 + g_1g_2)\vartheta^{013} \\
&\qquad -(2f_1g_3 + g_1f_3 + g_1g_3)\vartheta^{012}\Big)
\end{aligned}
$$

Thus

$$
\boxed{{}^{(8)}Q_i = e^{-2g}\Big(\big((\nabla f \cdot \nabla g) + (\nabla g)^2\big)\eta_{ik} - 2f_ig_k - g_if_k - g_ig_k\Big) * \vartheta^k} \qquad \text{(A.28)}
$$

References

[1] Cartan, E., "Geometry of Riemannian spaces", Math. Sci. Press, New York, 1983, p.217-231.

[2] A.Einstein:"Auf die Riemann-Metrik und den Fernparallelismus gegründete einheitliche Feldtheorie", Math. Ann. **102**, (1930), 658-697.

[3] A.Einstein, W.R.P. Mayer,: "Zwei strenge statische Lösungen der Feldgleichungen der einheitlichen Feldtheorie", *Sitzungsber. preuss, Acad. Wiss.,phys.-math.* **Kl** (1930) 110–120.

[4] R. Weitzenböck: "Invariantentheorie", Noordhoff, Groningen, 1923.

[5] E. Witten, S.-T. Yau: "Connectedness of the boundary in the AdS/CFT correspondence", hep-th/9910245.

[6] F. W. Hehl: "Four lectures on Poincaré gauge theory", in Proceedings of the 6-th Course on Spin, Torsion and Supergravity, held at Erice, Italy , 1979, eds. P.G. Bergmann, V. de Sabbata (Plenum, N.Y.) p.5

[7] T. Kawai, "Poincaré gauge theory of (2+1)- dimensional gravity", *Phys.Rev.*, **D49**, (1994) 2862-2871.

[8] F. Gronwald and F. W. Hehl: "On the gauge aspects of gravity". In: *International School of Cosmology and Gravitation:* 14 Course: Quantum Gravity, held May 1995 in Erice, Italy. Proceedings. P.G. Bergmann et al.(eds.) (World Scientific, Singapore 1996) pp. 148–198. Los Alamos eprint archive gr-qc/960201.

[9] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman: "Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance", *Physics Reports* **258**, (1995) 1–171.

[10] W. Kopczynski, "Problems with metric-teleparallel theories of gravitation", *J. Phys.* **A15**, (1982), 493.

[11] F. Müller-Hoissen, J.Nitsch: "Teleparallelism - A viable theory of gravity", *Phys.Rev.*, **D28**, (1983) 718.

[12] W. Thirring: *Classical Field Theory.* A Course in Mathematical Physics 2, 2nd ed. (Springer, New York, 1986).

[13] J. W. Maluf, "Sparling two-forms, the conformal factor and the gravitational energy density of the teleparallel equivalent of general relativity ", *Gen.Rel.Grav.* **30**, (1998) 413-423

[14] J. W. Maluf, J. F. da Rocha Neto: "General relativity on a null surface: Hamiltonian formulation in the teleparallel geometry", Los Alamos eprint archive gr-qc/9808001.

[15] R. S. Tung, J. M. Nester: "The quadratic spinor Lagrangian is equivalent to the teleparallel theory", Los Alamos eprint archive gr-qc/9809030

[16] J. M. Nester, H-J Yo: "Symmetric teleparallel general relativity", Los Alamos eprint archive gr-qc/9809049

[17] S. Kaniel and Y. Itin: "Gravity on a parallelizable manifold", *Nuovo Cim.* **113B** (1998), 393-400.

[18] U. Muench, F. Gronwald, F. W. Hehl: " A small guide to variations in teleparallel gauge theories of gravity and the Kaniel-Itin model", *Gen. Rel. Grav.*, **30**, (1998) 933-961.

[19] Y. Itin "Gravity on a parallelizable manifold. Exact solutions", *Gen. Rel. Grav.* **31** (1999) 187–204.

[20] Y. Itin, S. Kaniel "On a class of invariant coframe operators and application to gravity", gr-qc/9907023.

[21] Y. Itin: "On variations in teleparallelism theories",gr-qc/9904030.

[22] Yu. N. Obukhov: "The geometrical approach to antisymmetric tensor field theory", *Phys. Lett.* **109B** (1982) 195–199.

[23] Yu. N. Obukhov: "Gauge invariance in the theory of antisymmetric tensor fields", *Theor. Math. Phys.* **50** (1982) 229–235.

[24] F. Gronwald: "Metric-affine gauge theory of gravity I. Fundamental structure and field equations", *Int. J. Mod. Phys.* **D6** (1997) 263–303.

[25] E.W. Mielke: Ashtekar's complex variables in general relativity and its teleparallelism equivalent. *Ann. Phys. (N.Y.)* **219** (1992) 78–108.

[26] H. Rumpf: "On the translational part of the Lagrangian in the Poincaré gauge theory of gravitation", *Z. Naturf.* **33a** (1978) 1224–1225.

[27] P. von der Heyde: "Is gravitation mediated by the torsion of spacetime?", *Z. Naturf.* **31a** (1976) 1725–1726.

[28] J. Nitsch and F.W. Hehl: "Translational gauge theory of gravity: Post–Newtonian approximation and spin precession", *Phys. Lett.* **B90** (1980) 98–102.

[29] L. Landau, E. Lifshitz: *Theory of Field* (Oxford, Pergamon Press, 1962).